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Various Types of Five Dimensional Warp Factor and Effective Planck Scale

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Abstract

Based on the assumption that the warp factor of four dimensional space-time and the one of fifth dimension are tied through a parameter α, we consider five dimensional gravity with a 3-brane coupled to a bulk scalar field. For arbitrary value of α, the form of the warp factor is implicitly determined by hypergeometric function. Concretely, we show that the warp factor becomes explicit form for appropriate value of α, and study the relation between four dimensional effective Planck scale and the brane tension. This setup allows the possibility of extending the diversity of brane world.

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1 Introduction

After Horăva-Witten model in eleven dimensional SUGRA theory with two branes on S^1/Z_2 were proposed [1, 2], the path for the study of brane world scenario opened rapidly. Especially, in the framework of five dimensional theory with warp factor [3, 4, 5, 12] (warped metric function) which is exponential damping function of fifth dimension compactified on S^1/Z_2 orbifold, Randall and Sundrum suggested that the origin of the hierarchy problem comes from the warped geometry of the extra dimension. This scenario appeals to the possibility of an extra dimension limited by two 3-branes with tensions of opposite sign [3]. Moreover, if fifth dimension is not compactified, massless graviton is localized on the brane and this setup has usual four dimensional gravity law at large distance [4].

The Randall-Sundrum scenario including bulk scalar field has been proposed [6, 7, 8, 9, 10]. Identifying the scalar field with dilaton field, the scenario is inspired by string theory, sequentially, the solutions to the cosmological constant problem are very actively being investigated. In recent papers, the self-tuning mechanism of the cosmological constant has been suggested [6, 7]. The self-tuning idea allows a flat space solution without fine-tuning between input parameters in lagrangian. Thus, it is expected that the Randall-Sundrum scenario has the possibilities of making a breakthrough in particle physics or cosmology, and we are very interested in the form of the warp factor because the setup of the scenario is based on the ansatz for metric.

In this paper, we point out that the form of the warp factor has various types in the framework of five dimensional gravity with a 3-brane coupled to a bulk scalar field. As for metric, we make the assumption that the warp factor of four dimensional spacetime and the one of fifth dimension are related through a parameter α defined in ref.[13]. This metric corresponds to an extension of metric in the original Randall-Sundrum model and a new path will open the study of warp factor which is solution of Einstein equations. As a result of investigation, for arbitrary value of α, the form of warp factor with dependence of fifth dimension is implicitly determined by hypergeometric function [13]. Moreover, for particular value of α, we show that the explicit form of the warp factor and scalar field have variety. Under the assumption of infinite fifth dimension, it is important to study whether effective Planck scale is finite or not.

This paper is organized as follows. In section 2, we describe an action of the model and an ansatz for metric considered here. Solving the equations of motion, we obtain general form of warp factor controlled by both a parameter α and the sign of bulk cosmological constant. In section 3, we explore the warp factor for four cases, where $\alpha = 0, 8, 12$ and vanishing cosmological constant are described, separately. For each case, we compute the effective Planck scale, brane tension and the relation between these. A conclusion is given in final section.

2 The Model

We consider the following action

$$S = \int d^5x \sqrt{-G} \left\{ \frac{1}{2\kappa_5^2} \mathcal{R} - \frac{1}{2} (\nabla\phi)^2 - \Lambda \right\} + \int d^4x \sqrt{-g} \; \{ -f(\phi) \} \, , \qquad (1)$$

where Λ is the cosmological constant in the bulk and $1/\kappa_5^2 = M_*^3$, M_* is the fundamental scale of five dimensional theory. Here G is the five dimensional metric and g is the induced four dimensional metric on the brane which is located at $y = 0$, where y is the coordinate of fifth dimension. In the second term, $f(\phi)$ represents a brane tension coupled to a scalar field. We take the ansatz for metric [13]

$$\begin{aligned} ds^2 &= e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{2\alpha A(y)} dy^2 \\ &\equiv G_{MN} dx^M dx^N \, , \end{aligned} \qquad (2)$$

where $g_{\mu\nu} = diag(-,+,+,+)$. Note that the warp factor with y-dependence of four dimensional spacetime and the one of fifth dimension part are tied through a parameter α.

With the metric, the equations of motion are given by

$$A'' + (2-\alpha)(A')^2 = -\frac{\kappa_5^2}{3} \left\{ \frac{1}{2} (\phi')^2 + e^{2\alpha A} \Lambda \right\} - \frac{\kappa_5^2}{3} e^{\alpha A} f(\phi) \delta(y) \, , \qquad (3)$$

$$(A')^2 = \frac{1}{12} \kappa_5^2 (\phi')^2 - \frac{\kappa_5^2}{6} e^{2\alpha A} \Lambda \, , \qquad (4)$$

$$(4-\alpha) A' \phi' + \phi'' = e^{\alpha A} \frac{\partial f}{\partial \phi} \delta(y) \, , \qquad (5)$$

where the prime represents the derivative with respect to the y. From Eqs.(4) and (5), the equations in the bulk are expressed as

$$\phi' = c \, e^{(\alpha-4)A} \, . \qquad (6)$$

$$A' = \epsilon \frac{\sqrt{3}}{6} \kappa_5 \, |c| \, e^{(\alpha-4)A} \sqrt{1 - \frac{2\Lambda}{c^2} e^{8A}} \, , \qquad (7)$$

Hence c is the integration constant and $\epsilon = \pm$, where the sign ϵ determines the branch of the square root.

From Eqs.(6) and (7), we obtain the solution for a scalar field

$$\phi(y) = \epsilon \frac{c}{|c|} \frac{2\sqrt{3}}{\kappa_5} A(y) + k \qquad (8)$$

for $\Lambda = 0$, and

$$\phi(y) = -\epsilon \frac{c}{|c|} \frac{\sqrt{3}}{2\kappa_5} \tanh^{-1} \sqrt{1 - \frac{2\Lambda}{c^2} e^{8A(y)}} + l \qquad (9)$$

for $\Lambda \neq 0$. Here k and l are the integration constants. Consequently, $A(y)$ is expressed as [13]

$$e^{(4-\alpha)A}\ {}_2F_1\left(\frac{1}{2}, \frac{4-\alpha}{8}; \frac{12-\alpha}{8}; \frac{2\Lambda}{c^2}e^{8A}\right) = \epsilon\frac{\sqrt{3}}{6}\kappa_5\,|c|\,(4-\alpha)\,(y+d)\ , \qquad (10)$$

for $\alpha \neq 4, 12$. Here d is the integration constant and we can take normalization condition $A(0) = 0$. In the case of $\alpha = 4, 12$, since we cannot use the integral representation of hypergeometric function due to vanishing of second or third argument, $A(y)$ can be directly obtained by the integration of Eq.(7). The case of $\alpha = 4$ was investigated in ref.[13], we obtained the warp factor without having a singularity and showed that the bulk cosmological constant is bounded by both finite effective Planck scale and five dimensional fundamental scale. Mathematically, adopting specific value of α, the hypergeometric function can be described explicitly in terms of elementary function[†]. As mentioned later, we describe concrete form of warp factor.

Due to delta function in Eqs.(3) and (5), the jump conditions with respect to the first derivative of A and ϕ can be obtained. We get the following results for the jump conditions using Eqs.(3) and (5)

$$\begin{aligned} \epsilon_+\frac{1}{\tilde{d}_+} - \epsilon_-\frac{1}{\tilde{d}_-} &= \frac{\partial f}{\partial \phi}(\phi(0))\,, \\ \epsilon_+\sqrt{\frac{1}{\tilde{d}_+^2} - 2\Lambda} - \epsilon_-\sqrt{\frac{1}{\tilde{d}_-^2} - 2\Lambda} &= -\frac{2}{\sqrt{3}}\kappa_5 f(\phi(0))\,, \end{aligned} \qquad (11)$$

where

$$\frac{1}{\tilde{d}_\pm} = \frac{6}{\sqrt{3}\kappa_5(4-\alpha)}\ {}_2F_1\left(\frac{1}{2}, \frac{4-\alpha}{8}; \frac{12-\alpha}{8}; \frac{2\Lambda}{c_\pm^2}\right)\frac{1}{d_\pm}\,. \qquad (12)$$

We denote the sign and the integration constants for the positive region ($y > 0$) by $\epsilon_+\,, c_+\,, d_+$ and those for the negative region ($y < 0$) by $\epsilon_-\,, c_-\,, d_-$. Using the above equations, we can obtain the brane tension at $y = 0$.

Furthermore, if the fifth dimension is assumed to have Z_2 symmetry, we can impose the condition $\epsilon_+ = -\epsilon_-$ and $c_+ = -c_- = c$. Integrating out the fifth dimension, the four dimensional effective Planck scale is given by

$$\begin{aligned} M_{\rm Pl}^2 &= \frac{1}{\kappa_5^2}\left|\int dy\ e^{(2+\alpha)A(y)}\right| \\ &= \frac{\sqrt{3}}{2\kappa_5^3|c|}\left|\left[t^6\ {}_2F_1\left(\frac{3}{4}, \frac{1}{2}; \frac{7}{4}; \frac{2\Lambda}{c^2}t^8\right)\right]_{t=1}^{t=e^{A_+(r_c)}}\right|\,, \end{aligned} \qquad (13)$$

[†]Examples of the hypergeometric function represented by elementary function : ${}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-z^2\right) = \frac{\sinh^{-1}z}{z}$, ${}_2F_1\left(\frac{1}{2},-\frac{1}{2};\frac{1}{2};z^2\right) = \cos\left(\sin^{-1}z\right)$, ${}_2F_1\left(a,b;c;0\right) = 1$

where A_+ corresponds to the function for positive region ($y > 0$). From the second line in Eq.(13), the effective Planck scale is represented not in terms of α but in terms of A. Thus, the finite of effective Planck scale is determined by both the sign of Λ and the value of warped metric function at $y = r_c$. In this setup, since we consider the case of infinite fifth dimension, the limit of $r_c \to \infty$ must be taken. Below, we investigate the form of warp factor and the finite of effective Planck scale for four specific value of α.

3 Solutions

We study the two cases of $\alpha = 0, 8$ where the hypergeometric function in Eq.(10) is represented by elementary function, and the case of $\alpha = 12$ where hypergeometric function cannot be used, and the case of vanishing bulk cosmological constant, separately. Our setup is based on the assumption that the fifth dimension is not compactified with Z_2 symmetry. For four cases, we obtain the warp factor and estimate the brane tension ($V = f(\phi(0))$) and four dimensional effective Planck scale $M_{\rm Pl}$.

3.1 Case I

For $\alpha = 0$ and $\Lambda < 0$, from footnote †, we can obtain the warp factor as follows

$$e^{4A_\pm} = \frac{|c|}{\sqrt{2|\Lambda|}} \sinh\left(\pm \frac{2\sqrt{6|\Lambda|}}{3} \kappa_5 \ (\ y \pm d \ ) \right) , \tag{14}$$

where plus (minus) corresponds to the function for $y > 0$ ($y < 0$) region. The condition $A(0) = 0$ yields

$$\sinh \frac{2\sqrt{6|\Lambda|}}{3} \kappa_5 d = \frac{\sqrt{2|\Lambda|}}{|c|} . \tag{15}$$

Moreover, the jump condition at $y = 0$ leads to the relation between the brane tension V and the bulk cosmological constant Λ

$$V = -\frac{\sqrt{3(c^2 + 2|\Lambda|)}}{\kappa_5} . \tag{16}$$

From Eq.(13), we get

$$M_{\rm Pl}^2 = \frac{2\sqrt{3}}{3\kappa_5^3 |c|} \left| \left(\frac{|c|}{\sqrt{2|\Lambda|}} \right)^{\frac{3}{2}} s^{\frac{3}{2}} \ F\left(-s^2\right) - F\left(-\frac{2|\Lambda|}{c^2}\right) \right| , \tag{17}$$

where ${}_2F_1\left(\frac{3}{4},\frac{1}{2};\frac{7}{4};z\right) \equiv F(z)$ and we defined

$$s = \sinh\frac{2\sqrt{6|\Lambda|}}{3}\kappa_5(r_c + d)\,. \tag{18}$$

Since infinite fifth dimension corresponds to the limit of $s \to \infty$, Eq.(17) leads infinite effective Planck scale. Namely, in order to obtain the finite of effective Planck scale, it is necessary to consider the setup including another 3-brane at $y = r_c$ with finite fifth dimension. Furthermore, from Eq.(9), the scalar field is infinite everywhere due to negative cosmological constant in the bulk. Therefore, this case is excluded. The case of $\alpha = 0, \Lambda > 0$ had been already investigated [10, 11].

3.2 Case II

For $\alpha = 8$ and $\Lambda > 0$, from footnote † and Eq.(10), we have

$$e^{8A_\pm} = \left[\,\frac{2\Lambda}{c^2} + \frac{4}{3}k_5^2c^2(\,y \pm d\,)^2\,\right]^{-1}\,. \tag{19}$$

Imposing the condition $A(0) = 0$, we obtain the condition for the integration constants

$$\frac{2\Lambda}{c^2} + \frac{4}{3}k_5^2d^2c^2 = 1\,, \tag{20}$$

and the jump condition at $y = 0$ yields the brane tension

$$V = 2c^2d\,. \tag{21}$$

From the above equations, the integration constant d can be eliminated as follows

$$2\Lambda + \frac{1}{3}\kappa_5^2V^2 = c^2\,. \tag{22}$$

Thus, the brane tension is given by

$$V = \pm\frac{\sqrt{3(c^2 - 2\Lambda)}}{\kappa_5}\,, \tag{23}$$

therefore, the cosmological constant in the bulk is bounded

$$0 < \Lambda \le \frac{c^2}{2}\,. \tag{24}$$

In this case, effective Planck scale is expressed as

$$M_{\rm Pl}^2 = \frac{2\sqrt{3}}{3\kappa_5^2|c|}\left|\, u^{-\frac{3}{4}}F\left(\frac{2\Lambda}{c^2u}\right) - F\left(\frac{2\Lambda}{c^2}\right)\right|, \tag{25}$$

where we define u by

$$u = \frac{2\Lambda}{c^2} + \frac{4}{3}\kappa_5^2 c^2 (r_c + d)^2 \,. \tag{26}$$

Infinite fifth dimension ($u \to \infty$) yields finite effective Planck scale

$$M_{\rm Pl}^2 = \frac{2\sqrt{3}}{3\kappa_5^3|c|}F\left(\frac{2\Lambda}{c^2}\right). \tag{27}$$

Using Eq.(24) and the fact that $0 < F(z) \le \frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} \sim 1.8$ for $0 < z \le 1$,

$$M_{\rm Pl}^2 \sim \frac{M^{\frac{9}{2}}}{|c|} \,. \tag{28}$$

From Eq.(23), the bulk cosmological constant becomes

$$2\Lambda \sim \frac{M_*^9}{M_{\rm Pl}^4} - \frac{V^2}{3M_*^3}\,, \tag{29}$$

and the condition $\Lambda > 0$ leads to the following inequality

$$M_* > V^{\frac{1}{6}} M_{\rm Pl}^{\frac{1}{3}} \,. \tag{30}$$

Thus, effective Planck scale is finite and the lower bound of five dimensional fundamental scale is described in terms of both the brane tension and the effective Planck scale. Eqs.(29) and (30) are as same as the case of $\alpha = 4$ in ref.[13]. Moreover, using Eqs.(9) and (19), the scalar field in the bulk is described in terms of y.

3.3 Case III

For $\alpha = 12$, since third argument of the hypergeometric function in Eq.(10) is zero, we directly integrate Eq.(7). The solution is given by

$$\tanh^{-1}\sqrt{1 - \frac{2\Lambda}{c^2}e^{8A_\pm}} + \frac{c^2}{2\Lambda}e^{-8A_\pm}\sqrt{1 - \frac{2\Lambda}{c^2}e^{8A_\pm}} = \pm\frac{2\sqrt{3}k_5|c|^3}{3\Lambda}(y \pm d)\,, \tag{31}$$

using $A(0) = 0$, we get

$$\tanh^{-1}\sqrt{1 - \frac{2\Lambda}{c^2}} + \frac{c^2}{2\Lambda}\sqrt{1 - \frac{2\Lambda}{c^2}} = \frac{2\sqrt{3}k_5|c|^3}{3\Lambda}d\,. \tag{32}$$

Note that this solution only makes sense when the argument of inverse function of hyperbolic tangent is smaller than unity, namely,

$$0 < \Lambda \leq \frac{c^2}{2} \,. \tag{33}$$

From the above equations, the brane tension is expressed as

$$V = \frac{\sqrt{3(c^2 - 2\Lambda)}}{k_5} \,. \tag{34}$$

In infinite fifth dimension, from Eq.(31), we get $e^{A(\infty)} = 0$. Taking account of the range of Λ in Eq.(33), the effective Planck scale is expressed as

$$M_{\rm Pl}^2 \sim \frac{1}{\kappa_5^3 |c|} = \frac{M_*^{\frac{9}{2}}}{|c|} \,. \tag{35}$$

This result is as same as the case of $\alpha = 8$. Namely, the relation between effective Planck scale and brane tension is Eq.(29). Using Eqs.(9) and (31), the relation between bulk scalar field and y can be obtained.

3.4 Case IV

For $\Lambda = 0$ and $\alpha \neq 4$, we obtain the following warp factor

$$A_{\pm}(y) = \frac{1}{4-\alpha} \ln \left[\pm \frac{|4-\alpha|}{2\sqrt{3}} \kappa_5 |c| (\, y \pm d \,) \right] , \tag{36}$$

therefore, using the condition $A(0) = 0$, the brane tension is expressed as

$$V = \pm \frac{\sqrt{3}|c|}{\kappa_5} \,, \tag{37}$$

where the sign $+$ $(-)$ corresponds to the case of $\alpha > 4$ $(\alpha < 4)$.

The effective Planck scale is

$$M_{\rm Pl}^2 = \frac{\sqrt{3}}{2\kappa_5^3 |c|} \left| \, v^{\frac{6}{4-\alpha}} - 1 \, \right| , \tag{38}$$

where $v \equiv \dfrac{\kappa_5 |c|}{2\sqrt{3}} |4-\alpha| (r_c + d)$. In infinite fifth dimension ($v \to \infty$), the finite of effective Planck scale is determined by the sign of exponent of v in Eq.(38). Namely, for $\alpha < 4$, this case must be excluded because the effective Planck scale is infinite. However, for $\alpha > 4$, the effective Planck scale is finite

$$M_{\rm Pl}^2 \sim \frac{M_*^6}{V} \,, \tag{39}$$

where we used the result of Eq.(37). Obviously, vanishing brane tension leads infinite effective Planck scale. If the brane tension is TeV scale, five dimensional fundamental scale becomes

$$M_* \sim 10^8 \text{ GeV} \tag{40}$$

In addition, using Eqs.(8) and (36), the scalar field in the bulk is represented in terms of logarithmic function. This behavior is similar to the result of ref.[7].

4 Conclusion

We have presented various patterns of warp factor in brane world with a bulk scalar field. Under the assumption that the warp factor $e^{B(y)}$ of fifth dimension is related with the one $e^{A(y)}$ of four dimensional spacetime: $B = \alpha A$, we were able to find solutions of equations of motion for specific value of α. This setup corresponds to an extension of Randall-Sundrum type.

Based on a model with infinite fifth dimension, for four cases where $\alpha = 0, 8, 12$ as well vanishing bulk cosmological constant, we obtained the warp factor and brane tension, and investigated whether effective Planck scale is finite or not. The case of $\alpha = 0, \Lambda < 0$ is excluded because of infinite effective Planck scale, in addition, bulk scalar field is infinite everywhere. In the case of $\alpha = 8, \Lambda > 0$, warp factor has fractional form, and finite effective Planck scale is obtained. It is shown that bulk cosmological constant is bounded by integration constant, and five dimensional fundamental scale has lower bound. In the case of $\alpha = 12$, finite Planck scale, bulk cosmological constant is bounded, and brane tension becomes positive. In the case of $\alpha \neq 4, \Lambda = 0$, both warp factor and scalar field have logarithmic form. The case of $\alpha < 4$ is excluded due to infinite effective Planck. For $\alpha > 4$, we pointed out that effective Planck scale is finite, however, it is infinite if brane tension is zero. Thus, we explored the diversity of the five dimensional warped geometry.

There are articles which we discuss in future. It is necessary to study the massless gravitational fluctuations about our classical solution obtained here. Moreover, we are interested in cosmological evolution for each type of warp factor. This setup can be possibly applied to several scenarios. We will turn to these works elsewhere.

References

[1] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven dimensions," Nucl. Phys. B **460**, 506 (1996), hep-th/9510209.

[2] P. Horava and E. Witten, "Eleven-Dimensional Supergravity on a Manifold with Boundary," Nucl. Phys. B **475**, 94 (1996), hep-th/9603142.

[3] L. Randall and R. Sundrum, "A large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.

[4] L. Randall and R. Sundrum, "An alternative to compactification," Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.

[5] V. A. Rubakov and M. E. Shaposhnikov, "Extra Space-Time Dimensions: Towards A Solution To The Cosmological Constant Problem," Phys. Lett. B **125**, 139 (1983).

[6] S. Kachru, M. Schulz and E. Silverstein, "Bounds on curved domain walls in 5d gravity," Phys. Rev. D **62**, 085003 (2000), hep-th/0002121.

[7] S. Kachru, M. Schulz and E. Silverstein, "Self-tuning flat domain walls in 5d gravity and string theory," Phys. Rev. D **62**, 045021 (2000), hep-th/0001206.

[8] C. Csaki, J. Erlich, C. Grojean and T. Hollowood, "General properties of the self-tuning domain wall approach to the cosmological constant problem,"Nucl. Phys. B **584**, 359 (2000), hep-th/0004133.

[9] P. Binetruy, J. M. Cline and C. Grojean, "Dynamical instability of brane solutions with a self-tuning cosmological constant," Phys. Lett. B **489**, 403 (2000), hep-th/0007029.

[10] H. Collins and B. Holdom, "The Randall-Sundrum scenario with an extra warped dimension," Phys. Rev. D **64**, 064003 (2001), hep-ph/0103103.

[11] H. Collins and B. Holdom, "The cosmological constant and warped extra dimensions," Phys. Rev. D **63**, 084020 (2001), hep-th/0009127.

[12] M. Ito, "Warped geometry in higher dimensions with an orbifold extra dimension," hep-th/0105186.

[13] M. Ito, "Five dimensional warped geometry with bulk scalar field,", hep-th/0109040.